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Tetraplectic structures, tri-momentum maps, and quaternionic flag manifolds

Philip Foth

Department of Mathematics, University of Arizona, Tucson, AZ 85721-0089, USA

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Abstract

The purpose of this note is to define tri-momentum maps for certain manifolds with an $Sp(1)^n$-action. We exhibit many interesting examples of such spaces using quaternions. We show how these maps can be used to reduce such manifolds to ones with fewer symmetries. The images of such maps for quaternionic flag manifolds, which are defined using the Dieudonné determinant, resemble the polytopes from the complex case. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

A closed form on a manifold can be used to gather information about its geometry [10]. In particular, when a manifold is equipped with a closed non-degenerate two-form ω, the manifold can be effectively studied by the methods of symplectic geometry. If, in addition, a Lie group acts on the manifold in a hamiltonian fashion, the Marsden–Weinstein reduction allows one to reduce the system to another one with fewer degrees of freedom.

Higher order closed differential forms with properties similar to the symplectic structures (e.g. zero characteristic distribution) are called multisymplectic forms and have important applications to field theories, like in Tulczyjew [34], Marsden et al. [25], Cantrijn et al. [2], and other works.

In this paper we concentrate on four-forms, and our main spaces of interest are quaternionic vector spaces and quaternionic flag manifolds. The main reason is that these spaces carry natural interesting group actions, and appear quite naturally in many different instances. For example, quaternionic flag manifolds can be realized as $Sp(n)$-orbits on the

E-mail address: foth@math.arizona.edu (P. Foth).

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space of quaternionic hermitian matrices. We show that these spaces carry natural closed non-degenerate invariant four-forms.

Our results are related to Yang–Mills theory. On quaternionic Kähler manifolds, Taniguchi [33] has established explicit C^0 neighbourhoods of the minimal Yang–Mills fields, which contain no other Yang Mills fields up to gauge equivalence. The quaternionic Yang–Mills connections ∇ are those that satisfy $d_\nabla(F^\nabla \wedge \psi) = 0$, where ψ is a four-forms as above.

In general, if X is an oriented $4m$-dimensional manifold equipped with a closed, non-degenerate four-forms ψ, then we call (X, ψ) a *tetraplectic* manifold. If, in addition, X is equipped with a $Sp(1)^n$ action satisfying certain properties (see Section 3), we can define a *tri-momentum map* from X to $(\wedge^3\mathfrak{s}^*)^n$, where $\mathfrak{s} = \mathfrak{sp}_1$. Under certain conditions, we can reduce the original manifold X to another manifold, which also possesses a tetraplectic structure. This procedure is quite different from the Hyper–Kähler reduction of Hitchin et al. [18] and the quaternionic reduction defined by Galicki and Lawson [14], since our target space is different.

We show how our procedure can be applied to the case when X is a quaternionic vector space or a (full or partial) quaternionic flag manifold. In particular, in the latter case, the images of the tri-momentum maps resemble the momentum map polytops for the torus actions on the complex flag manifolds. The coordinate expressions for the tri-momentum maps for the quaternionic flag manifolds can be obtained using the Dieudonné determinant [6].

In Section 5 of the paper we discuss some related developments.

2. Tetraplectic structures

Definition 2.1. Let X be a real manifold of dimension $4m$ and let ψ be a four-forms on X satisfying the following two conditions.

1. The form ψ is closed: $\mathrm{d}\psi = 0$.
2. The $4m$-form ψ^m is the volume form on X.

If these two conditions are satisfied, we call ψ a tetraplectic structure on X, and (X, ψ) a tetraplectic manifold.

One of the properties of ψ, that is an immediate consequence of the definition, is that the induced maps

$$\tilde{\psi}_x : T_x X \to \wedge^3 T_x^* X,$$

$$v \mapsto i_v \psi_x$$

have trivial kernels. One finds a large class of examples of tetraplectic structures given by the symplectic manifolds. If (X, ω) is a $4m$-dimensional symplectic manifold, then $\psi = \omega \wedge \omega$ is a tetraplectic structure on X. However, this class of manifolds will be of little interest to us, since such manifolds can be effectively treated by the methods of symplectic geometry. More interesting examples that we have in mind include quaternionic vector spaces $\mathbb{H}^m$, full and partial quaternionic flag manifolds, and manifolds with $(\mathbb{H}^*)^n$-action. Many of the manifolds in these examples do not allow symplectic structures.

Example. The first and basic example that we have in mind is the space $X = \mathbb{H}^m$. If we identify this space with $\mathbb{R}^{4m}$ in the usual way, the standard tetraplectic form ψ is defined by

$$\psi = \sum_{i=1}^{n} \mathrm{d}x_{4i-3} \wedge \mathrm{d}x_{4i-2} \wedge \mathrm{d}x_{4i-1} \wedge \mathrm{d}x_{4i},$$

where $x_1, \dots, x_{4m}$ is the coordinate system on $\mathbb{R}^{4m}$. The usual identification between $\mathbb{H}^m$ with coordinates $(q_1, \dots, q_m)$ and $\mathbb{R}^{4m}$ is given by

$$q_i = x_{4i-3} + \boldsymbol{i}x_{4i-2} + \boldsymbol{j}x_{4i-1} + \boldsymbol{k}x_{4i}.$$

We note that the form ψ is not the square of a symplectic form on $\mathbb{R}^{4m}$ for $m > 1$, because the square of a symplectic form would induce isomorphisms $\wedge^2 T_x X \simeq \wedge^2 T_x^* X$, obtained by contraction. However, in our case, when $m > 1$, one can easily see that there will be a non-trivial kernel at any point. Therefore, a naive attempt to obtain local Darboux coordinates for every tetraplectic structure fails. Later we will discuss a certain condition on ψ which will help to get a canonical local form. We would like to mention in this regard that in [2], the authors describe certain canonical models of multisymplectic structures.

Example. The first important compact example of a tetraplectic structure is given by the four-sphere S^4, which can also be viewed as the quaternionic projective line. The tetraplectic form ψ on S^4 is just a volume form. Under the identification with $\mathbb{HP}^1 \simeq Sp(2)/(Sp(1) \times Sp(1))$, we can choose an $Sp(2)$-invariant volume form. For example, if we represent S^4 as $\mathbb{H}$ plus the North pole, such a form would be given by

$$\psi = \frac{|q|^3}{(1+|q|^4)^2}\, \mathrm{d}|q|\Omega,$$

where we identify $\mathbb{H}^*$ with $\mathbb{R}_+ \times Sp(1)$, and let $|q|$ be the absolute value of $q \in \mathbb{H}$ and Ω an invariant volume three-form on $Sp(1)$. (We will always view $Sp(1)$ as the group of unit length quaternions.) Notice that S^4 allows neither a complex nor a symplectic structure. We also obtain a natural $Sp(1) \times Sp(1)$ action on $\mathbb{HP}^1$ coming from the natural $Sp(1) \times Sp(1)$ action on $\mathbb{H}^2$.

Example. Let us recall the large class of quaternionic Kähler manifolds, which are $4m$-dimensional Riemannian manifolds with the holonomy group a subgroup of $Sp(n) \times Sp(1)$. It was shown by Kraines [21] that all quaternionic Kähler manifolds are tetraplectic. However, this class does not exhaust our interest, because only quaternionic projective spaces are quaternionic Kähler, and not grassmannians or general flag manifolds. An interested reader should consult a beautiful survey by Salamon [29] and references therein.

Example. A particularly large class of four-dimensional tetraplectic manifolds is given by the Kulkarni four-folds [23], which come naturally endowed with a canonical conformal class of locally conformally flat metrics [35]. One can view these four-folds as quaternionic analogues of Riemann surfaces. These manifolds with their volume forms play an important role in the four-dimensional conformal field theory.

Definition 2.2. A spheroid Σ^n is the n-fold product of $Sp(1) \simeq S^3$, viewed as a Lie group.

The Lie algebra σ_n of Σ^n is the direct sum of n copies of $\mathfrak{s} = \mathfrak{sp}_1$—the maximal compact subalgebra of $\mathfrak{gl}(1, \mathbb{H})$. The Lie algebra $\mathfrak{s}$ can be identified with $\mathfrak{so}(3)$ and with $\mathbb{R}^3$, where the Lie bracket is the cross-product of two vectors (the vector product).

There are natural actions of spheroids on $\mathbb{H}^n$ and other interesting spaces. This will be our main motivation for Section 3.

3. Tri-momentum maps and reduction

Let X be a $4m$-manifold equipped with a tetraplectic structure given by a four-forms ψ. Let a spheroid Σ^n act on X preserving the form ψ (by tetraplectomorphisms). The stabilizer of a point $x \in X$ is not necessarily a spheroid. For example, if one considers the product $S^2 \times \mathbb{R}^2$, and the action of Σ^1 via $SO(3)$ on the first factor, then there exists a volume form, which is not changed by this action, and a stabilizer of a point is a circle. All our further examples will be such that the stabilizer of are actually spheroids, and we will tacitly bear in mind this assumption for the general discussion as well.

If Σ acts on (X, ψ) as above, then we have a canonical map

$$\sigma \to \Gamma(X, TX)$$

sending an element Z of σ to a vector field $\tilde{Z}$ on X. Then we also have the map $\sigma \to A^3(X)$ given by $Z \mapsto i_{\tilde{Z}}\psi$. If a three-form given by $i_Y\psi$, where Y is a vector field on X, is closed, then we call Y a *locally hamiltonian vector field*. If, in addition, $i_Y\psi$ is exact, then we call Y simply a *hamiltonian vector field*. Here, one can speculate that the group $H^3(X, \mathbb{R})$ can be viewed as a certain topological obstruction.

Consider the four-vector field ξ on X uniquely defined by $i_\xi\psi^m = \psi^{m-1}$. This four-vector field defines a quaternary operation $\{\cdot, \cdot, \cdot, \cdot\}$ on $C^\infty(X)$ in a standard fashion. If the Schouten bracket of ξ with itself happens to vanish (which is the case for all our applications), then we get a generalized Poisson algebra structure on $C^\infty(X)$ in the terminology of [4]. In the same source, as well as in [19], the authors consider triples of functions $f_1, f_2, f_3 \in C^\infty(X)$ and the corresponding hamiltonian vector fields given by

$$Y_{f_1,f_2,f_3} = i(\mathrm{d}f_1 \wedge \mathrm{d}f_2 \wedge \mathrm{d}f_3)\xi.$$

Then the corresponding evolution equation for any $g \in C^\infty(X)$ is given by

$$\dot{g} = \{f_1, f_2, f_3, g\}.$$

We say that Σ acts on X in a (generalized) hamiltonian way if each of the generating vector fields for the action is hamiltonian. The dual vector space to the Lie algebra σ_n is isomorphic to the product of n copies of $\mathfrak{s}^*$. The action of Σ^n on X induces a map $(\wedge^3\mathfrak{s})^n \to \Gamma(X, \wedge^3 TX)$ by taking the third exterior power of the the above morphism for each component Σ_1 and adding these up.

Definition 3.1. Let Σ^n act on (X, ψ) in a generalized hamiltonian way. A tri-momentum map μ is a map

$$\mu : X \to (\wedge^3 \mathfrak{s}^*)^n \simeq \mathbb{R}^n$$

satisfying the following conditions.

1. μ is Σ^n-invariant: $\mu(a \cdot x) = \mu(x)$, for $a \in \Sigma^n$.
2. For any $\delta \in (\wedge^3 \mathfrak{s})^n$ we have

   $$\mathrm{d}(\mu(x), \delta) = i_{\tilde{\delta}} \psi,$$

   where $x \in X$, and $\tilde{\delta}$ is the tri-vector field on X induced by δ.
3. For any $x \in X$, such that $\mu(x)$ is regular, Ker $T_x\mu = (\wedge^3 \sigma \cdot x)^{\perp}$ with respect to ψ_x.

Notice that the first statement in the above definition is equivalent to saying that μ is Σ^n-equivariant, because the co-adjoint action of Σ^1 on $\mathfrak{s}^*$ induces the trivial action of Σ^1 on $\wedge^3 \mathfrak{s}^*$.

We will always identify $(\wedge^3 \mathfrak{s}^*)^n$ with $\mathbb{R}^n$ unless it leads to confusion. The following is an example of a tri-momentum map. Other examples will be treated later on.

Example. Let $X = (\mathbb{H}^n, \psi)$ as in Section 2 with the standard spheroid action. The tri-momentum map $\mathbb{H}^n \to \mathbb{R}^n$ is given by

$$(q_1, \ldots, q_n) \to (|q_1|^4, \ldots, |q_n|^4).$$

The level sets for this tri-momentum map are isomorphic to the products of three-spheres.

Example. Let us take $X = \mathbb{H}^2$ and the diagonal action of $a \in \Sigma^1$ on $(q_1, q_2) \in \mathbb{H}^2$ given by $(a \cdot q_1, a \cdot q_2)$. Then the tri-momentum map $\mathbb{H}^2 \to \mathbb{R}$ is given by $(q_1, q_2) \to |q_1|^4 + |q_2|^4$. The regular level sets for this tri-momentum map are isomorphic to seven-spheres.

Now we would like to define the procedure of reduction in the general setup of tri-momentum maps. Let $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ be a regular point of a tri-momentum map $\mu : (X, \psi) \to \mathbb{R}^n$ as above. The level set $Z_{\boldsymbol{x}} := \mu^{-1}(\boldsymbol{x})$ is smooth and Σ^n-invariant. The stabilizers of the points in $Z_{\boldsymbol{x}} \subset X$ form a group bundle over it, which we assume to be smooth. Then the reduced space $Y_{\boldsymbol{x}} := Z_{\boldsymbol{x}}/\Sigma^n$ is well defined and is a smooth manifold. Let us also assume that ψ is *horizontal* on $Z_{\boldsymbol{x}}$, meaning that for any $\beta \in \sigma_n$, and the corresponding vector field $\tilde{\beta}$ on $Z_{\boldsymbol{x}}$, one has $i_{\tilde{\beta}} \psi_{|Z_{\boldsymbol{x}}} = 0$ (one can easily see that a priori, the four-forms on the level sets need not necessarily be horizontal). By methods similar to those used for the symplectic reduction [26], we prove the following.

Theorem 3.1. *Let $\boldsymbol{x} \in \mathbb{R}^n$ be a regular value of a tri-momentum map $\mu : X \to \mathbb{R}^n$. Assume that the stabilizers of all points in $Z_{\boldsymbol{x}}$ form a smooth spheroid bundle over $Z_{\boldsymbol{x}}$, and that ψ* it is horizontal on *$Z_{\boldsymbol{x}}$. Then the reduced space $Y_{\boldsymbol{x}} = X//\Sigma^n$ corresponding to $\boldsymbol{x}$ is a smooth manifold admitting a tetraplectic structure $\psi_{\boldsymbol{x}}$, which is reduced from ψ.*

Proof. The tetraplectic structure ψ on X induces a totally anti-symmetric four-linear form on each of the spaces $T_z Z_{\boldsymbol{x}}/T_z(\Sigma.z)$. This form is well defined due to the invariance and horizontality of the form ψ. Therefore, we have a global four form $\psi_{\boldsymbol{x}}$ on the reduced space

$Y_{\boldsymbol{x}}$. Now let us show that for any $y \in Y_{\boldsymbol{x}}$, the induced map $T_y Y_{\boldsymbol{x}} \to \wedge^3 T_y^* Y_{\boldsymbol{x}}$ has trivial kernel. It is enough to work with the case of $n = 1$, i.e. the $Sp(1)$-action. Since the actions of different summands commute, and are hamiltonian, the reduction can be performed one step at a time. In this case, one can choose a non-zero element in $\wedge^3 \mathfrak{s}$, which would define a Bott–Morse function $f(z)$ on the manifold X satisfying $\mathrm{d}f = \alpha$, where α is the one-form, obtained by contracting the generating tri-vector field on X for the Σ^1 action with ψ. According to our definition, α is the generating one-form for the codimension one foliation determined by f (which is regular, locally near the regular level sets). Therefore, we can represent the volume form ψ^m as the product $\mathrm{d}f \wedge \beta \wedge \Omega$, where β is an invariant three-form, which pairs non-trivially with the fundamental three-vector field, and Ω is an invariant $(4m - 4)$-form, which reduces to $Y_{\boldsymbol{x}}$ and is the highest exterior power of the tetraplectic form $\psi_{\boldsymbol{x}}$. □

Example. We leave the majority of examples for the subsequent sections, and consider only the two examples that we had earlier in this section.

In the first example, when $X = \mathbb{H}^n$ with the standard tetraplectic form ψ and the standard Σ^n action, the reduced spaces are just points.

In a slight modification of our second example, let $X = \mathbb{H}^2$ with the standard diagonal Σ^1 action, and let the four-forms ψ be given by

$$\psi = \mathrm{d}(|q_1|^4 - |q_2|^4) \wedge \mathrm{d}(\alpha_1 - \alpha_2) \wedge \mathrm{d}(\beta_1 - \beta_2) \wedge \mathrm{d}(\gamma_1 - \gamma_2),$$

where $(q_1, q_2) \in \mathbb{H}^2$, and q_i has the absolute value $|q_i|$ and the spherical part $(\alpha_i, \beta_i, \gamma_i)$. The reduced space in this example is isomorphic to $\mathbb{HP}^1 \simeq S^4$, and topologically we have the Hopf fibration $S^3 \to S^7 \to S^4$. The reduced tetraplectic structure on $\mathbb{HP}^1$ is just the invariant volume form discussed in Section 2. Similarly, one can obtain an invariant tetraplectic structure on $\mathbb{HP}^n$ for an arbitrary n.

We would like to reiterate that the reduction procedure described above is different from the Hyper–Kähler reduction [18]and quaternionic reduction [14]. The group that acts in our situation is $Sp(1)^n$ and the target for the momentum map involves third exterior powers of the Lie algebra summands. Whereas, for example in [14], the groups maybe different, but the momentum mapping is bundle valued.

We remark that one can obtain focal sets $\mathrm{Foc}_{\mathbb{HP}^n}\mathbb{CP}^n$ (critical sets for of the normal exponential map with respect to the totally geodesic submanifold $\mathbb{CP}^n \subset \mathbb{HP}^n$) as zero level sets of a particular momentum map as in Ornea and Piccinni [27]. It would be interesting to see if one can obtain new examples of Sasakian–Einstein structures using our tri-momentum maps.

4. Quaternionic flag manifolds

In this section, we show that the classical constructions of (full and partial) complex flag manifolds can be used to construct quaternionic flag manifolds using the reduction procedure that we discussed in Section 3. Moreover, we show that the reductions of these spaces possess natural invariant tetraplectic structures. Let $G = GL(n, \mathbb{H})$ and B be the subgroup of upper triangular matrices. One has a natural identification between the full flag manifold $F_n :=$

$Sp(n)/\Sigma^n$ and G/B similar to the complex case. We also consider the partial flag manifolds $F_{i_1,\dots,i_j} := Sp(n)/(Sp(i_1) \times \cdots \times Sp(i_j))$, where $n = i_1 + \cdots + i_j$, where for example, we have the quaternionic grassmannians $\mathrm{Gr}(p, n-p) = Sp(n)/(Sp(p) \times Sp(n-p))$ appearing as conjugacy classes in the classical compact group $Sp(n)$. They are the orbits of the elements $\mathrm{diag}(\underbrace{1,\dots 1}_{p}, \underbrace{-1,\dots,-1}_{n-p})$, when $Sp(n)$ is considered as a matrix subgroup of G. The advantage of our approach is that although the lack of determinants over skew fields does not allow one to use the Plücker determinants for the quaternionic flags, the tri-momentums maps still exist and have certain nice properties which we will exhibit.

Let us consider the space $\mathcal{H}_n$ of quaternionic $n \times n$ hermitian matrices, defined as a subspace of $n \times n$ quaternionic matrices $\mathfrak{gl}_n(\mathbb{H})$ by the condition $A = A^*$, where A^* stands for the transposed quaternionic conjugate matrix. The group $Sp(n)$ acts by conjugation on $\mathcal{H}_n$ and the orbits of the action are isomorphic to quaternionic flag manifolds.

The cell decomposition enumerated by the Schubert symbols works over $\mathbb{H}$ as well as it does over $\mathbb{C}$ [8]. One can also use an identification of $\mathbb{H}^n$ with $\mathbb{R}^{4n}$ and embed quaternionic flag manifolds into the real ones in order to construct a non-degenerate Morse function on $F_{i_1,\dots,i_j}$, essentially done in [28].

A very interesting question related to the space $\mathcal{H}_n$ was discussed by Fulton [13]. It turns out that the equation $A_1 + \cdots + A_n = C$, where the matrices have prescribed spectra, has a solution in quaternionic hermitian matrices if and only if it has a solution in complex hermitian matrices.

First of all, we notice that the grassmannian $\mathrm{Gr}(p, n-p)$ can be realized as follows. Consider the space $\mathbb{H}^{np}$ of $n \times p$ matrices with quaternionic entries, and let the subspace $V \subset \mathbb{H}^{np}$ consist of those matrices whose rows are orthonormal with respect to the standard pairing:

$$\langle \xi, \eta \rangle = \sum_{i=1}^{n} \xi_i \bar{\eta}_i,$$

where the bar stands for the quaternionic conjugation. The group $Sp(p)$ acts on such bases preserving V, on which it acts freely. The quotient space is isomorphic to the quaternionic grassmannian $\mathrm{Gr}(p, n-p)$. We claim that there exists a tetraplectic form ψ on $\mathbb{H}^{np}$, that can be pulled back to V. The resulting four-forms on V will be preserved by the action of $Sp(p)$ and horizontal, and thus, will descend to the quotient, $\mathrm{Gr}(p, n-p)$. This would endow the irreducible symmetric space $\mathrm{Gr}(p, n-p)$ with a tetraplectic structure.

More generally, we can extend the construction of Guillemin and Sternberg [16] for the complex flags. Since the full flag projects to all partial flags, and this projection behaves well with the respect to the group action, a tetraplectic structure on the full flag manifold, F_n, would push down to the partial flags.

Let $K = Sp(n)$ and let Σ^n be its maximal spheroid as defined in Section 2. We have a principal fibration $K \to F_n$ with fiber Σ^n. We need the following projection map:

$$\mu_2 : K \times \mathbb{R}^n \to \mathbb{R}^n,$$

$$k \times x \mapsto x.$$

Here, we identified for convenience $\mathbb{R}^n$ with $(\wedge^3\mathfrak{s}^*)^n$ (see Section 3). The map μ_2 is the projection to the second factor, where we use $\sigma_n = \underbrace{\mathfrak{s} \oplus \cdots \oplus \mathfrak{s}}_{n}$ to take third exterior powers of individual summands. The principal bundle $K \to F_n$ admits an invariant bundle-valued three form, δ, from which we obtain a pre-tetraplectic four-forms:

$$\psi_1 = d\langle \delta, \mu_2 \rangle$$

on $K \times \mathbb{R}^n$ (cf. minimal coupling [32]).

Following the strategy of [16] we can show that the form ψ_1 is actually tetraplectic, when we restrict to the proper subspace $\mathbb{R}^n_0$ of $\mathbb{R}^n$ (using the above identification we can actually let $\mathbb{R}^n_0 = (\mathbb{R}_+)^n$). We call this restriction ψ. Now the map μ_2 which we call μ after restricting it to $K \times \mathbb{R}^n_0$, has all the properties of a tri-momentum map from Section 3. For a generic $\xi \in (\mathbb{R}_+)^n \subset \mathbb{R}^n$, it is clear that Σ^n stabilizes ξ. Therefore, we obtain the following result.

Proposition 4.1. *The action of Σ^n on $K \times \mathbb{R}^n_0$, where Σ^n acts trivially on the second factor, has the tri-momentum map μ. The reduced space is isomorphic to F_n, the full quaternionic flag manifold. The reduced tetraplectic form on F_n so obtained is K-invariant.*

Let us outline the relationship of the four-forms ψ on the quaternionic flag manifolds that we have obtained with invariant symplectic structures on other K-homogeneous spaces. Let T be a maximal torus in K contained in Σ^n and let $T_i \simeq S^1$ be a maximal torus in the ith component of Σ^n, so that $T = T_1 \times \cdots \times T_n$. We have the following fibration:

$$\prod_{i=1}^{n} \left(\frac{\Sigma^1}{T_i} \right) \to \frac{Sp(n)}{T} \to F_n.$$

Each factor Σ^1/T_i is isomorphic to $\mathbb{CP}^1$ and carries an Σ^1-invariant symplectic form ω_i, while the space $Sp(n)/T$ is the classical flag manifold isomorphic to $Sp(n, \mathbb{C})$ modulo its Borel subgroup, and thus, carries a K-invariant symplectic form ω (actually, this form can be obtained, once one identifies K/T with a co-adjoint orbit and uses the Kirillov–Kostant–Souriau (KKS) structure on the later). Trivially, by choosing a fiber, we can assume that all the ω_i are the pull-backs of the form ω. The spectral sequence for this fiber bundle clearly shows that, cohomologically, one can choose such a K-invariant tetraplectic structure ψ on F_n that it will correspond to the cohomology class of $\omega \wedge \omega$. Moreover, due to the K-invariance of the aforementioned, this correspondence can be traced on the level of forms.

At this point, we would like to construct canonical four-forms on all the orbits of $Sp(n)$ action on $\mathcal{H}_n$. These orbits, as we mentioned earlier, are isomorphic to quaternionic flag manifolds. These forms have similar origins and properties to the KKS symplectic forms on the coadjoint orbits of the group $U(n)$. First of all, let us define a four-commutator of square matrices:

$$[A_1, A_2, A_3, A_4] = \sum_{\tau \in S_4} \operatorname{sign}(\tau) A_{\tau(1)} A_{\tau(2)} A_{\tau(3)} A_{\tau(4)}.$$

This four-commutator has the following property with respect to the usual commutator:

$$\sum_{1\leq i<j\leq 5}(-1)^{i+j}[[A_i, A_j], A_1, \dots, \hat{A}_i, \dots, \hat{A}_j, \dots, A_5] = 0. \tag{4.1}$$

We also notice that the four-commutator of four quaternionic hermitian matrices is again such, so we have an operation $\mathcal{H}_n^{\otimes 4} \to \mathcal{H}_n$. Let us also recall the non-degenerate pairing $\mathcal{H}_n \times \mathcal{H}_n \to \mathbb{R}$ given by the real part of the trace of the product: $(A, B) \to Re\,Tr(AB)$. This pairing is invariant with respect to $K = Sp(n)$ action. This allows us to identify the tangent and co-tangent space to any element $y \in \mathcal{H}_n$ with $\mathcal{H}_n$. The four-vector field κ on $\mathcal{H}_n$, defined via the above four-vector field is parallel to the orbits. The corresponding four-forms ψ on an orbit $\mathcal{O}$, whose value at $y \in \mathcal{O} \subset \mathcal{H}_n$ is given by

$$\psi_y(A_1, A_2, A_3, A_4) = Re\,Tr(y[A_1, A_2, A_3, A_4])$$

has the following properties.

Proposition 4.2. *The four-forms ψ is non-degenerate, closed, and $Sp(n)$-invariant.*

Proof. The invariance is a direct consequence of the fact that the real part of the trace of the product of two quaternionic matrices is conjugation invariant. Non-degeneracy is easy to check at one point of the orbit, namely, the diagonal matrix. Then one can use the invariance to show non-degeneracy on the whole orbit. To show that ψ is closed, we will follow discussion on p. 229 of [20]. We will identify the orbit $\mathcal{O}$ with K/L, where L stabilizes y, and use the fact that K-invariant four-forms on K/L correspond uniquely to L-invariant elements in $\wedge^4(\mathfrak{l}^{\perp})$, where the differential is given by the formula:

$$\mathrm{d}\phi(X_1, \dots, X_5) = \frac{1}{5}\sum_{i<j}(-1)^{i+j+1}\phi([X_i, X_j], \dots, \hat{X}_i, \dots, \hat{X}_j, \dots).$$

Therefore, the closedness immediately follows from our formula (4.1). □

Thus, we have shown that the quaternionic flag manifolds, which appear as $Sp(n)$ orbits in $\mathcal{H}_n$, are naturally tetraplectic. [1]

Now we will discuss some general properties of the momentum polytopes and we will see how the classical polytopes for the Hamiltonian torus actions on complex flag manifolds fit into the quaternionic picture. The group $H = (\mathbb{H}^*)^n$ acts on $\mathbb{H}^n$ and this action induces, in turn, an action of H on the spaces such as F_n and $\mathrm{Gr}(p, n-p)$. The maximal spheroid Σ^n is always thought of as the maximal compact subgroup of H.

Recall the Dieudonné determinant [6]:

$$D : GL(n, \mathbb{H}) \to \mathbb{R}_+,$$

which is defined using the transformation of a matrix to an upper-triangular form. For example, when $n = 1$, $D(q) = |q|$, the usual norm. For any $A \in GL(n, \mathbb{H})$ of

[1] I was informed that Reyer Sjamaar and Yi Lin have a different construction of tetraplectic four-forms on quaternionic flag manifolds, using natural Lie algebra valued differential forms and tautological vector bundles.

the form:

$$\begin{pmatrix} q_1 & H \\ 0 & B \end{pmatrix},$$

where H is any row vector of length $(n-1)$, and B is an $(n-1)\times(n-1)$ matrix from $GL(n-1,\mathbb{H})$, the Dieudonné determinant of A is given by

$$D(A) = |q_1|\cdot D(B).$$

Among the properties of the Dieudonné determinant are many of the usual properties of the determinant in the group $GL(n)$ over a commutative field. We will use the Dieudonné determinant D to construct a tri-momentum map for the quaternionic grassmannians $\mathrm{Gr}(p,n-p)$.

First of all, it is well known that the combinatorics of the (partial) flag manifolds in the quaternionic case is the same as in the complex case. Moreover, the cohomology rings are isomorphic (with the appropriate change in grading), and the Shubert calculus works the same.

Let us first treat the case of the quaternionic grassmannian $X = \mathrm{Gr}(p,n-p)$. We choose a K-invariant tetraplectic four-forms ψ on X, which is really only defined up to multiplication by a scalar. The spheroid Σ^n acts on X in a tetraplectomorphic way preserving ψ. We claim that the image of the tri-momentum map is the same as in the complex case, i.e. can be identified with the polytope Z_p^n in $\mathbb{R}^n$ defined by

$$Z_p^n := \{0 \le x_i \le 1, x_1+\cdots+x_n = p\},$$

where $(x_1,\ldots,x_n)$ are the coordinates in $\mathbb{R}^n$. One of the ways of looking at the coordinates $x_1,\ldots,x_n$ is that of viewing them as the hamiltonians for the actions of the summands of Σ^n, and we claim that the three-vector field δ_i, determined by the ith summand in $\Sigma^n = \oplus_{i=1}^n \Sigma^1$, satisfies $\mathrm{d}x_i = i_{\delta_i}\psi$.

Now the construction of the coordinate function x_i on $\mathrm{Gr}(p,n-p)$ is not really different from the complex case. Any quaternionc p-plane Π in $\mathbb{H}^n$ can be viewed as an $n\times p$ matrix M_Π of rank p with quaternionic entries. Following [15], let J be a subset of $\{1,\ldots,n\}$ of cardinality p. By $M_\Pi(J)$ we understand the $p\times p$ matrix with quaternionic entries obtained from M by keeping only those rows that are numbered by the elements of J. We further define:

$$x_i(\Pi) = \frac{\sum_{i\in J} D^4(M_\Pi(J))}{\sum_J D^4(M_\Pi(J))}.$$

We note that these Dieudonné determinants and their properties are of a crucial use in the quaternionic case.

Proposition 4.3. *The coordinates* $\{x_i\}$ *give a tri-momentum map*

$$\mu : \mathrm{Gr}(p,n-p) \to \mathbb{R}^n.$$

The image of this map is the convex polytope $Z_p^n \subset \mathbb{R}^n$. *The vertices of the polytope* Z_p^n *are the points in* $\mathrm{Gr}(p,n-p)$ *fixed under the* Σ^n*-action. If* Π *is a point in* $\mathrm{Gr}(p,n-p)$, *then* *the image of the closure of the orbit* $\mu(\overline{\Sigma.\Pi})$ *is the convex hull of the images of the fixed points in* $\overline{\Sigma.\Pi}$.

Proof. The main idea is to choose an identification between $\mathrm{Gr}(p, n-p)$ and an orbit of $Sp(n)$ in $\mathcal{H}_n$, say of the element $\mathrm{diag}(\underbrace{0,\ldots,0}_{n-p},\underbrace{1,\ldots,1}_{p})$. For example, $\mathbb{HP}^1$ can be identified with an orbit of $Sp(2)$ of diag(0, 1). The element of this orbit of the form

$$\begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} s_1 & s_2 \\ s_3 & s_4 \end{pmatrix}^{-1} = \begin{pmatrix} |s_2|^2 & s_2\bar{s}_4 \\ s_4\bar{s}_2 & |s_4|^2 \end{pmatrix},$$

where the first matrix is from $Sp(2)$, and the second from $\mathcal{H}_2$, corresponds to $[s_2 : s_4] \in \mathbb{HP}^1$. On the other hand, a point $\mathbb{HP}^1$ can be represented as a line in $\mathbb{H}^2$ passing through a point (q_1, q_2). There are certain formulas relating q_1, q_2 with s_1, s_3, which can be obtained by a straightforward computation. Analogous considerations are valid for all $\mathrm{Gr}(p, n-p)$. One the orbit side, the momentum map will simply be given by the projection to the diagonal. □

Similarly, one can obtain statements about full and all partial quaternionic flag manifolds that are analogous to the complex case.

5. Related developments

In this section, we will merely outline the content of two subsequent papers [12] and [11]. In [12], we show several important generalizations of the classical results from symplectic geometry to the case of tetraplectic geometry. The first is the convexity theorem founded in [1] and [17]. Basically, we establish the following fact. Let (X, ψ) be a tetraplectic manifold and let Σ^n act on X in a tetraplectomorphic way. Let $(f_1, \ldots, f_n)$ be such functions on X that the flow corresponding to the generalized hamiltonian three-vector fields $\delta_1, \ldots, \delta_n$ (defined by $i_{\delta_j}\psi = \mathrm{d}f_j$) generated a subgroup of $\mathrm{Diff}(X)$ defined by Σ^n. Then the image of the map $\mu : X \to \mathbb{R}^n$ given by

$$\mu(\boldsymbol{x}) = (f_1(\boldsymbol{x}), \ldots, f_n(\boldsymbol{x}))$$

is the convex hull of the images of connected components of the set of common critical points of f_i.

Another direction that we pursued [12] is a generalization of the Duistermaat–Heckman theorems [7]. For the case of quaternionic flag manifolds and certain other compact manifolds with $(\mathbb{H}^*)^n$ action, this formula would help to recover the structure of the cohomology ring of the reduced spaces from the combinatorics of the fixed point data combined with a generalization of Duistermaat–Heckman by methods similar to Guillemin and Sternberg [16].

In [11], we work with generalized Poisson structures (GPS) of rank 4 as defined by de Ascárraga et al. [4]. We show that many familiar manifolds have natural GPS. In particular, we show that the full quaternionic flag manifolds F_n (as well as all the partial ones) have interesting natural GPS. In particular, we give a Lie theoretic construction of the Bruhat four-vector field on F_n, which is an example of GPS. Recall that the classical Bruhat Poisson structures on complex flag manifolds that were first introduced by Soibelman [30] and independently by Lu and Weinstein [24]. One of their main properties is that that the symplectic leaf decomposition yields exactly the Bruhat cells. Moreover, Evens and Lu [9] showed that the Kostant harmonic forms [22] have Poisson harmonic nature with respect to the Bruhat Poisson structure. It would be interesting to find a basis in cohomology of $H^\bullet(F_n)$ dual to the natural Bruhat cell decomposition for quaternionic flag manifolds F_n that is represented by forms with properties similar to Kostant harmonic forms. In particular, establish a relationship with the Bruhat four-vector field of [11]. It would also appear quite natural to consider the equivariant cohomology with the respect to the natural spheroid action on F_n.

We also plan to study analogues of other interesting facts from the complex geometry, which can be adapted to the quaternionic case. Examples include the Gelfand–MacPherson correspondence between GIT and symplectic quotients of grassmannians and products of projective spaces, the Gelfand–Tsetlin coordinates on the space of hermitian matrices, moduli spaces of quaternionic vector bundles, and others.

5.1. *Manifolds with* $(\mathbb{H}^*)^n$*-action*

One could be tempted to use the theory of the toric manifolds to study the manifolds with an $(\mathbb{H}^*)^n$ action with a dense open orbit. In particular, one can start with a convex polytope in $\mathbb{R}^n$ and try to construct a $4n$-dimensional manifold with an $(\mathbb{H}^*)^n$ action that has a dense open orbit such that the tri-momentum map for the corresponding Σ^n-action is that convex polytope. We do not know if this is possible in general except the simplest situation, when the polytope is the standard simplex in $\mathbb{R}^n$. In this case, the corresponding manifold is $\mathbb{HP}^n$. A naive application of the usual reduction method of constructing such manifolds fails in general due to the non-commutative nature of Σ^1. However, there are examples of classes of manifolds on which $(\mathbb{H}^*)^n$ acts with a dense open orbit and we believe that one can classify those by methods similar to [3] and [5]. Scott [31] develops a theory of quaternionic toric manifolds, using topological methods. In general, only a single copy of $Sp(1)$ acts on those. If we only assume that $\mathbb{H}^*$ acts on a manifold X, then in certain cases one can obtain a cell decomposition of X similar to the Bialynicki–Birula decomposition in the complex case. Examples of such spaces are given by the quaternionic flag manifolds.

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